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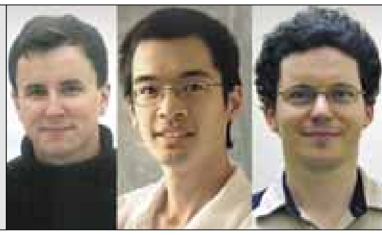

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Bruno de Finetti – a great probabilist and a great man

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Bruno de Finetti was born in 1906. One hundred years later, in the year 2006, the birth of this mathematician is celebrated everywhere in the (mathematical) world through a series of meetings, events and conferences. In Italy the Accademia Nazionale dei Lincei, the Mathematics Department of the University of Roma "La Sapienza" and the Unione Matematica Italiana (UMI) are going to put in action a series of initiatives. The UMI, for instance, aims to promote a very big event in September and also to publish two volumes in the series "Opere dei Grandi Matematici" in which a Selecta of the most important papers of de Finetti will be collected together. In the first volume there will be about forty papers concerning the scientific work that de Finetti did in probability and statistics; in the second there will be work that de Finetti did in economics, financial science, actuarial science, mathematical analysis, and (mathematical) education and popularization.

Biography

As a member of the committee appointed (by the UMI) to choose the papers of the Selecta, I would like to recall some facts relating to the life and scientific thinking of this famous mathematician.

Bruno de Finetti was born in Innsbruck to Italian parents; they were also Austrian citizens as he himself wrote in an autobiographic note accompanying the book [1] edited by his former students and friends on the occasion of his 75th birthday. In 1906 his father was working in Innsbruck as a railway constructor; he was an engineer as was his father before him. Thus it was no surprise when in 1923 Bruno de Finetti enrolled at Milano Polytechnic. There he discovered his true passion for mathematics and during his third year at Milano Polytechnic, perhaps inspired by a paper by the biologist Carlo Foà, he started research in the field of population genetics, which soon led him (aged twenty years) to the first of almost three hundred writings. It was the first example of a model with overlapping generations in population genetics and it was many years ahead of its time. Even today, bioscience researchers quote the results of the young de Finetti.

He then moved to the recently founded University of Milan and there, in 1927, he graduated in mathematics with a dissertation on affine geometry. Among his teachers at the University of Milan, it is worth mentioning Oscar Chisini, who is well known in statistics for his general definition of "Chisini's mean".

At the time de Finetti received his degree, a position was waiting for him in Roma at the Italian Central Statistical Institute, which was founded and directed by Corrado Gini. De Finetti remained there until 1931, after which he moved to Trieste and started working for a big insurance company. There he worked as an actuary and also on the mechanization of some actuarial services. In the following years, he supplemented his work with several academic appointments both in Trieste and Padova. In 1947 he became a full professor firstly at the University of Trieste and subsequently at "La Sapienza" University of Roma, where he remained until the end of his career.

Numerous letters, memoranda, newspaper clippings, articles and court documents give evidence of de Finetti's political and social activism. It is worth considering his vital interest in civil aspects and social justice [6]. His longing for social justice caused him in the 1970's to be a candidate in several elections and he was also arrested for his antimilitarist position. At the time of his death in 1985, Bruno de Finetti was an honorary fellow of the Royal Statistical Society as well as a member of the UMI, a member of the International Statistical Institute and a fellow of the Institute of Mathematical Statistics. Additionally, in 1974, he had been elected a corresponding member, and then a full member, of the Accademia dei Lincei. Many details on his life are given in the papers of M.D. Cifarelli and E. Regazzini [5] (in which a broad picture of the scientific milieu in which Bruno de Finetti took the first steps of his scientific career is given) as well as by L. Daboni [6], who was appointed by the UMI to the official commemoration for the Bollettino of the UMI (the society's major journal). Some significant flashes of the history of probability in Italy, in which de Finetti played the main role, are given in [7] and [8].

A summary of de Finetti's scientific contributions

Bruno de Finetti is known worldwide as one of the most important probabilists and statisticians of the 20th century. In fact, even in his former position in Roma, he was laying the foundations for his principal contributions to probability theory and statistics: the subjective approach to probability (i.e., the *operational subjective* conception of probability), the definition and analysis of sequences of exchangeable events, the definition and analysis of processes with stationary independent increments and infinitely decomposable laws, and the theory of mean values (it is worth remarking that in this period he qualified as a university lecturer of mathematical analysis; the examiners were Giuseppe Peano, Mauro Picone and Salvatore Pincherle). De Finetti started working on probability and statistics in a period of tremendous development for these subjects. For instance A. N. Kolmogorov and P. Lévy were giving their decisive contributions to the modern theory of probability and R.A. Fisher was setting out the basic technical concepts for his new approach to statistics. In Italy

At the 2nd Berkeley Symposium, from left to right: M. Loeve, P. Lévy, W. Feller, B. de Finetti

Guido Castelnuovo, Francesco Paolo Cantelli, Corrado Gini and young de Finetti became impressed by this cultural revival. Moreover a very big event took place over these years, i.e., the International Mathematical Union (IMU) Congress in Bologna. The former president of the UMI, Salvatore Pincherle, successfully worked, at the end of the world war, to get together all the people who had a keen interest in mathematics irrespective of nationality. So around 840 mathematicians assembled in Bologna, among them de Finetti and the most famous probabilists and statisticians of the time: Maurice Fréchet, Aleksandr Y. Khinchin, Paul Lévy, Jerzy Neyman, Ronald A. Fisher and George Pólya.

Let us summarize the main scientific contributions of Bruno de Finetti: Major research topics studied by de Finetti were probability (subjective theory, calculus, Bayes's theorem) and statistics but also mechanization, genetics, mathematical analysis, mathematics applied to economics (game theory, financial and actuarial mathematics), and popularization and educational mathematics.

Mechanization: as stated above, working in an insurance company probably contributed to making him one of the first mathematicians to be aware of the possibilities offered by computing machinery. Later on, after 1950, in the position of adviser to the Italian Research Council (CNR), he was instrumental in getting the first electronic computer to the INAC (National Institute for Applied Computation) in Roma, whose director at the time was Mauro Picone.

Genetics: even today a de Finetti diagram is used to graph the genotype frequencies of populations where there are two alleles and the population is diploid. It is based on an equilateral triangle and on the theorem that from any point within the triangle the sum of the lengths of the three lines from that point to the sides of the triangle, where these lines are perpendicular to the sides, is equal.

Mathematical analysis: at the beginning of his scientific career de Finetti studied the characteristic properties of vectorial analysis with regard to the case of projective homographies (bijective maps between linear spaces). Subsequently he considered some very important topics in mathematical analysis like measures in abstract spaces, the Riemann-Stieltjes integral and convex stratifications. The latter are now known as quasiconvex functions (or quasi concave as W. Fenchel named them later on). Since then convex and quasiconvex analysis has been widely applied in many fields such as optimization theory, game theory, and linear and nonlinear programming.

Economics: Bruno de Finetti's interest in economics was innate and led him during his first year at the Milano Polytechnic to attend the lectures given there by Ulisse Gobbi, who was later the dean of the important economics and financial studies at "Bocconi" University. The lectures, in turn, confirmed his radical position, which he himself summarised as follows in his autobiographical note [1]:

> ...the only directive of the whole of economics, freed from the damned game and tangle of individual and group egoisms, should always be the realisation of a collective Pareto optimum inspired by some criterion of equity.

Educational: it is worth emphasizing the devotion of such a great scientist to mathematical education topics. As Carla Rossi (one of de Finetti's pupils) said in [8] the substance of de Finetti's approach and ideas of teaching can be found in his every scientific paper even more than in the many works specifically devoted to that issue. And about problems to take into consideration he used to say

> ... before approaching a problem to solve you need to see it, in order that a subject of study, specifically Mathematics, does not appear sterile, obscure and useless, it should always be presented so that studying it is fully and genuinely justified.

Involvement in school reform and teaching methodology was one of his major interests throughout his life. A wide variety of materials illustrate de Finetti's efforts to improve science and mathematics teaching, teacher education, and school curricula, e.g., his writings *Il Saper vedere in matematica* (Know-how in Mathematics) and *Perchè la matematica?* (Why Mathematics?), his plans for educational movies, and a project for an educational centre for teachers.

Now let us shortly outline the major contributions given by de Finetti in the fields of probability and statistics theory.

Probability and statistics

The classic exposition of his distinctive theory are the papers [2] and [3], in which he discussed probability founded on the coherence of betting odds and the consequences of exchangeability. A "summa" of Bruno de Finetti's revolutionary ideas

Bruno de Finetti (1979)

on probability can be found in the two volumes [4] of his best known book *Teoria della Probabilità* (1970), which was translated into English in 1975. However, his contributions to probability and statistics do not reduce to his subjective approach; they include important results on finitely additive measures, processes with independent increments, sequences of exchangeable variables and associative means (see the review [5] for details on these).

(a) The concepts of probability in de Finetti's time: until the 1920's it was tacitly assumed that the frequentist interpretative ideas of probability played the main role in the various applications of the discipline, or at least the most popular methods of assessment were based on a combinatorial approach or on an observed frequency. The idea of subjective probability was almost surely undermined by the developments in physics after a few considerations on it given by a couple of French scientists (E. Borel (1924) and P. Lévy (1925); see [10] for a suitable reference).

Since his early days as a mathematician, Bruno de Finetti revitalized the theory of subjective probability in a very different spirit with respect to the past. De Finetti probabilism (as he called it in [2]) is the "*true heir of the empiricist philosophical tradition in the spirit of David Hume … de Finetti was a prodigy who could make his philosophical and conceptual ideas match his mathematical developments*" [10]. In an interesting correspondence with M. Fréchet in the 1930's de Finetti, discussing some papers concerning the almost sure convergence of a sequence of independent and identically distributed bernoullian random variables, states that the problems concerning stochastic convergence are mere signs of a deeper problem concerning the correct mathematical definition of probability. As a matter of fact, in his opinion, the definition has to adhere to the intuitive notion of probability as it is conceived by every one of us in usual everyday life. He maintains that one has no right to make arbitrary use of the properties introduced to give a mathematical definition of probability. Indeed, these very properties have to be not only formally consistent but also intrinsically necessary with respect to a meaningful interpretation of probability. De Finetti shares Fréchet's opinion implying that each concept, even of a mere mathematical nature, is more or less directly triggered by intuition. Nevertheless, this definition can effectively be arbitrary, provided that one confines oneself to deduce purely formal conclusions from it. This turns out to be the case in the definition of measure. A different case is connected with the definition of weight, since *we cannot force a pair of scales to work according to our definition*.

De Finetti proposed a thought experiment along the following lines (a philosophical gambling strategy): you must set the price of a promise to pay 1 (lira in de Finetti's time) if, for instance, there was life on Mars one billion years ago and 0 if there was not, and tomorrow the answer will be revealed. You know that your opponent will be able to choose either to buy such a promise from you at the price you have set, or require you to buy such a promise from them, still at the same price. In other words, you set the odds but your opponent decides which side of the bet will be yours. The price you set is the operational subjective probability that you assign to the proposition on which you are betting. This price has to obey the probability axioms if you are not to face certain loss, as you would if you set a price above 1 (or a negative price). It is seen that in any application of probability theory we can interpret the probabilities as personal degrees of belief of a rational agent; this is the term reserved for a person who will not accept a Dutch book. By considering bets on more than one event de Finetti could justify additivity. Prices, or equivalently odds, that do not expose you to certain loss through a Dutch book are called coherent. Probability will be the *degree of belief* assigned by you to the *occurrence* of an event.

The mathematical formulation of probability $\mathbb{P}$ was given in [2]. Given a class E of events and an element A of the class, any $p \in [0,1]$ represents a coherent assessment on A. After defining a probability de Finetti proves that the usual rules of the calculus of probability are necessary for the coherence of $\mathbb{P}$ on E, i.e., he states the well known properties (except σ-additivity):

If $\mathbb{P}$ is a probability on a class E, we have:

1. $A \in E \Longrightarrow \mathbb{P}(A) \in [0,1]$;
2. $\Omega \in E \Longrightarrow \mathbb{P}(\Omega) = 1$ (here Ω is the certain event);
3. if $A_1, \cdots, A_n \in E, \cup_{k=1}^{n} A_k \in E$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ then $\mathbb{P}(\cup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mathbb{P}(A_k)$.

These classical properties, i.e., the fact that $\mathbb{P}$ is a function whose range lies between 0 and 1 (these two extreme values being assumed by, but not kept only for, the *impossible* and the *certain* events respectively) and which is additive for mutually exclusive events, constitute the starting point in the axiomatic approach; so de Finetti can rightly claim that the subjective view can only enlarge and never restrict the practical purport of probability theory.

Subsequently, in 1949, with regard to the problem of existence of at least a probability on a given class of events, he provided the following extension theorem:

If A and B are classes of events such that $A \subset B$ and $\mathbb{P}_1$ is a probability on A, then there is a probability $\mathbb{P}_2$ on B such that $\mathbb{P}_1 = \mathbb{P}_2$ on A.

He also showed that the previous methods (i.e., the combinatorial and the frequentist methods) can be recovered if some useful (even if very particular) methods of coherent evaluation are considered; they are subjective as well and they are unnecessarily restricted to the domain of applicability. De Finetti also makes absolutely clear the distinction between the subjective character of the notion of probability and the objective character of the elements (i.e., events) to which it refers.

Although there is no reason why different interpretations (senses) of a word cannot be used in different contexts, there is a history of antagonism between the followers of de Finetti (sometimes called Bayesians) and frequentists, with the latter often rejecting the subjective interpretation as ill-grounded. The groups have also disagreed about which of the two senses reflects what is commonly meant by the term probable. In the preface of many books concerning probability theory there is a wide trace of this controversial dispute. Today the long wave of the subjective approach of de Finetti is growing more and more in the field of assessments of probability. All the work of de Finetti exhibits an intuitionist and constructivist view, with a natural bent for submitting the mathematical formulation of probability theory only to the needs required by any practical application.

(b) Stochastic processes with independent increments: the crisis of determinism and of the casuality principle introduces a novelty into the scientific method. Rigid laws stating that a certain fact is bound to occur in a certain way are being replaced by probabilistic or statistical laws stating that a certain fact can occur depending on a variety of ways governed by probability laws. Thus, given a scalar quantity whose temporal evolution is described by $X = X(t)$, $t \geq 0$, one assumes that the values taken by $X(t)$ are known for $t \leq t_0$ and considers the conditional increment $\{(X(t) - X(t_0)/X(u), u \leq t_0,\ t > t_0\}$. As far as the probability distribution function $F(\cdot)$ of such an increment is concerned, de Finetti considers the three cases:

1. $F(\cdot)$ is independent of $X(u)$ for every $u \in [0,t_0]$ - ($F(\cdot)$ is called known);
2. $F(\cdot)$ is independent of $X(u)$ for every $u \in [0,t_0)$ - ($F(\cdot)$ is called differential);
3. $F(\cdot)$ is dependent on $X(u)$ on $[0,t_0]$ - ($F(\cdot)$ is called integral).

De Finetti deals with the problem of characterizing the probability distribution of $X(t)$: if $X(0) = 0$ and ϕ_t, ψ_t denote the probability distribution function and the characteristic function of $X(t)$ respectively, then $\{\psi_{\frac{1}{n}}(\cdot)\}^n$ is the characteristic function of the sum of n independent increments, identically distributed according to the law $X(t) - X(0)$. In modern literature these processes are known as processes with homogeneous independent increments and $\psi_1[= (\psi_{\frac{1}{n}})^n]$ is called the infinitely decomposable characteristic function; de Finetti shows that ψ_t is continuous whenever X is continuous on $[0,+\infty)$ and $X(t)$ is different from ct. Moreover the examples chosen to emphasize the relevance of the continuity for X are very noteworthy: the Poisson process and the compound Poisson process. The method de Finetti uses here is quite innovative with respect to the past. Finally he achieves the well known result:

The class of infinitely decomposable laws coincides with the class of distributions limits of finite convolutions of distributions of Poisson type.

This result was a starting point for a subsequent series of papers by A. N. Kolmogorov and P. Lévy.

(c) Exchangeability: the works of P. Levy and G. Castelnuovo (from 1925 to 1928) taught him the analytical tools for arriving at one of the most important results in the theory of probability, i.e., the concept of exchangeability of events (1928), followed (in 1929) by the probability laws of continuous time random processes.

With regard to the connections between the subjective viewpoint and the objective one, which in a different way characterizes the classical approach and the frequentist approaches, these procedures are, according to de Finetti, not necessarily conducive to the existence of an objective probability. But, if the classical probability assignment can be justified immediately by judging the events equally probable, the analysis of the frequentistic point of view is more complex. To do that de Finetti broke the analysis down into two steps (explaining their subjective foundations): the first deals with the relations between the assignments of probabilities and the prevision of future frequencies; the second concerns the relationship between the observation of past frequencies and the prevision of future frequencies.

Let us consider a sequence of events $E_1, E_2, \cdots$ relative to a sequence of trials and suppose that, under the hypothesis H_N stating a certain result of the first N events, a person considers equally probable the events $E_{N+1}, E_{N+2}, \cdots$. Then, denoting by f_{H_N} the prevision of the random relative frequency of the occurence of n events $E_{N+1}, E_{N+2}, \cdots, E_{N+n}$ conditional to H_N, the well known properties of a prevision yield $p_{H_N} = f_{H_N}$, where p_{H_N} indicates the probability of each $E_{N+1}, E_{N+2}, \cdots$ conditional to H_N. But when is it possible to estimate f_{H_N} in such a manner? De Finetti's answer is: when the events considered are supposed to be elements of a stochastic process whose probability law, conditional on a large sample, admits, as prevision of the future frequencies, a value approximately equal to the frequency observed in these samples. Since the choice of the probability law governing the stochastic process is subjective, the prevision of a future frequency based on the observation of those past is naturally subjective. This procedure is perfectly admissible when the process is *exchangeable*, that is when only information about the number of successes and failures is relevant, irrespective of exactly which trials are successes or failures.

De Finetti defines a sequence of events to be *equivalent* (the word "exchangeable" was proposed later by Pólya) in a communication at the above mentioned IMU Congress of Bologna. Subsequently de Finetti was able to justify the evaluation of f_{H_N} via past frequencies thanks to some important representation theorems (see [5] for a suitable reference).

(d) The de Finetti–Kolmogorov–Nagumo theorem: de Finetti worked in the field of statistics firstly by approaching descrip-

tive statistics and afterwards inductive reasoning. We confine ourselves to the first argument leaving to the considerations illustrated in the previous section the main ideas of statistical inference. "Reasoning by induction" means, according to de Finetti's interpretation, learning from experience, and this thought provoking remark is clear enough and wholly pervasive.

In a paper of 1931 de Finetti obtained a significant extension of a theorem independently proved by Kolmogorov and Nagumo. To this aim he extends Chisini's definition of a mean to distribution functions in the following way: *given a class F of frequency distribution functions on $\mathbb{R}$ and a real valued function f on F, a mean of ϕ in F, with respect to the evaluation of f, is any number ρ such that $f(\phi) = f(D_\rho)$ where D_ρ denotes the probability distribution function which degenerates at x.*

Subsequently let $A, B, A < B$ be real numbers and let $F = F(A,B)$ denote the class of all distribution functions whose support is included in $[A,B]$; moreover he defines $m : F \to \mathbb{R}$ through $f(D_{m(\phi)}) = f(\phi)$ for any distribution $\phi \in F$. Finally the result is given:

Suppose that $m : F[A,B] \to \mathbb{R}$ is a consistent, strictly increasing and associative mean. Then there is a function ψ, continuous and strictly increasing in $[A,B]$, for which $m(\phi) = \psi^{-1}(\int_{\mathbb{R}} \psi(x) d\phi(x))$, $(\phi \in F(A,B))$. Moreover ψ is uniquely determined up to linear transformations. Conversely, if m is defined as before for a function ψ with the properties stated, then it satisfies consistency, strict monotonicity and associativity.

The latter properties of consistency, strict monotonicity and associativity identify with a well know definition, in terms of random gains and of stochastic dominance.

Final remark

It is strange that the summary of a lifetime of work on the theory of something should begin by the declaration that something does not exist but so begins de Finetti's Theory of Probability [4]: *My thesis, paradoxically, and a little provocatively, but nonetheless genuinely, is simply this: Probability does not exist.* This conveys his idea that probability is an expression of the observer's view of the world and as such it has no existence of its own. As a consequence of the subjective approach, statistical inference is no longer an empirical process producing opinions from data but it becomes a logical-psychological process selecting opinions compatible with data among the available ones. In de Finetti's theory, bets are for money, so your probability of an event is effectively the price that you are willing to pay for a lottery ticket that yields 1 unit of money if the event occurs and nothing otherwise; de Finetti used the (Italian) notation 'Pr' to refer interchangeably to Probability, Price and Prevision (foresight) and he treated them as alternative labels for a single concept. The appeal of his money based definition is that it has the same beauty and simplicity as theories of (modern) physics; the measurements are direct and operational, they involve exchanges of a naturally conserved quantity and their empirical laws are deducible from a single governing principle, namely the principle of coherence or non-arbitration. The coherence condition can also be shown to be very useful for welfare evaluations, where it provides a natural foundation for utilitarianism. Starting from [3], where the famous argument of de Finetti on decision under uncertainty is presented, economists tried to develop an argument stating a natural condition that turns out to imply the existence of coherent subjective probabilities and can justify a model of choice based on them; de Finetti's idea served later as a point of departure for Savage's theory of subjective expected utility (see [9]).

We conclude by quoting de Finetti himself.

> The only relevant thing is uncertainty – the extent of our knowledge and ignorance. The actual fact of whether or not the events considered are in some sense determined, or known by other people, and so on, is of no consequence.

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